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Bounded Möbius invariant $\mathcal{Q}_K$ spaces

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Abstract

We prove the corona theorem for the Banach algebra $\mathcal{Q}_K \cap H^\infty$ under some assumptions of the weight K, and a Fefferman–Stein type decomposition of the boundary values of $\mathcal{Q}_K$ is obtained.

Keywords: $\mathcal{Q}_K$ spaces; Corona problems; Carleson-type measures

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane and let $\mathbb{T}$ be the boundary of $\mathbb{D}$. Let H^∞ be the class of all bounded analytic functions on $\mathbb{D}$ with the supremum norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

For a function $K : [0, +\infty) \to [0, +\infty)$, consider the space $\mathcal{Q}_K$ of all analytic functions on $\mathbb{D}$ for which

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\big(g(z,w)\big)\, dA(z) < \infty,$$

where $dA(z)$ is the area measure on $\mathbb{D}$ and $g(z,w) = -\log|\varphi_w(z)|$ is the Green's function on $\mathbb{D}$ with pole at $w \in \mathbb{D}$, and $\varphi_w(z) = \frac{w-z}{1-\bar{w}z}$ is a Möbius transformation of $\mathbb{D}$. The space $\mathcal{Q}_K$ is Möbius invariant in the sense that

$$\|f \circ \varphi_a\|_{\mathcal{Q}_K} = \|f\|_{\mathcal{Q}_K}, \quad a \in \mathbb{D}.$$

We also consider the space $\mathcal{Q}_{K,0}$ of all functions $f \in \mathcal{Q}_K$ for which

$$\lim_{|w|\to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K\big(g(z,w)\big)\, dA(z) = 0.$$

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The special case when $K(t) = t^p$, $0 < p < \infty$, has been more intensively studied (see [10]). The corresponding spaces are usually denoted by $\mathcal{Q}_p$ and were introduced in [1]. If $p = 1$ then $\mathcal{Q}_1 = BMOA = BMO(\mathbb{T}) \cap H^2$, where H^2 is the classical Hardy space and $BMO(\mathbb{T})$ is the usual space of functions in $L^2(\mathbb{T})$ with bounded mean oscillation on $\mathbb{T}$. Also, for $p > 1$, the space $\mathcal{Q}_p$ turns to be the classical Bloch space $\mathcal{B}$ of all analytic functions on $\mathbb{D}$ for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} \bigl(1 - |z|^2\bigr)\bigl|f'(z)\bigr| < \infty.$$

It is clear that the function-theoretic properties of $\mathcal{Q}_K$ depend on the structure of K. So, like in [3], from now on we take it for granted that the above weight function K always satisfies the following conditions:

(a) K is nondecreasing;
(b) K is two times differentiable on $(0, 1)$;
(c) $\int_0^{1/e} K(\log(1/r))r\,dr < \infty$;
(d) $K(t) = K(1) > 0$, $t \geqslant 1$;
(e) $K(2t) \approx K(t)$, $t \geqslant 0$.

We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ such that $a \leqslant Cb$, and the notation $a \approx b$ (a is comparable with b) means that $a \lesssim b \lesssim a$. Some comments about these conditions: condition (a) ensures that each $\mathcal{Q}_K$ is a subspace of $\mathcal{B}$ (see [9]); (c) implies that $\mathcal{Q}_K$ is nontrivial, that is, in this case $\mathcal{Q}_K$ contains nonconstant functions; and condition (d) says that only the behaviour of K near the origen is important.

We also need two more conditions on K as follows:

$$\int_0^1 \varphi_K(s)\frac{ds}{s} < \infty \tag{1}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s^2} < \infty \tag{2}$$

where

$$\varphi_K(s) = \sup_{0\leqslant t\leqslant 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

We note that $K(t) = t^p$ for $0 < t \leqslant 1$ and $K(t) = K(1)$ for $t > 1$ satisfies conditions (a)–(e) and (1) for $0 < p < \infty$ and (2) for $0 < p < 1$.

This paper is a natural continuation of [3], extending some of the results of [7] from $\mathcal{Q}_p$ to $\mathcal{Q}_K$, and is principally devoted to the study of the Banach algebra $\mathcal{Q}_K \cap H^\infty$ ($\mathcal{Q}_{K,0} \cap H^\infty$). First of all, we will prove that the corona theorem holds for the algebra $\mathcal{Q}_K \cap H^\infty$ ($\mathcal{Q}_{K,0} \cap H^\infty$), whenever K satisfies conditions (1) and (2), that is, the unit disc $\mathbb{D}$ is dense in the maximal ideal space of $\mathcal{Q}_K \cap H^\infty$ ($\mathcal{Q}_{K,0} \cap H^\infty$). This fact can be reformulated in the following way.

Theorem 1.1. *Suppose that* (1) *and* (2) *hold for K. Let $f_1, \ldots, f_n \in \mathcal{Q}_K \cap H^\infty$ ($\mathcal{Q}_{K,0} \cap H^\infty$) with*

$$\inf_{z\in\mathbb{D}} \sum_{k=1}^n \bigl|f_k(z)\bigr| > 0.$$

Then there exist $g_1, \ldots, g_n \in \mathcal{Q}_K \cap H^\infty$ ($\mathcal{Q}_{K,0} \cap H^\infty$) such that

$$f_1 g_1 + \cdots + f_n g_n = 1.$$

In case that $BMOA \subset \mathcal{Q}_K$, the space $\mathcal{Q}_K \cap H^\infty$ is just H^∞, and the corona theorem holds by a famous result of L. Carleson [2]. For $K(t) = t^p$, $0 < p < 1$, Theorem 1.1 was proved by A. Nicolau and J. Xiao (see [7]). It is well known that there is a close connection between $\bar\partial$-equations and the Fefferman–Stein decomposition asserting that any function f in $BMO(\mathbb{T})$ can be decomposed into $f = u + \tilde v$, where $u, v \in L^\infty(\mathbb{T})$ and $\tilde v$ is the conjugate function of v. So, it is not surprising that solving $\bar\partial$-equations with some appropriate estimates leads to the following result.

Theorem 1.2. *Suppose that* (1) *and* (2) *hold for* K, *and let* $f \in L^2(\mathbb{T})$.

(i) $f \in \mathcal{Q}_K(\mathbb{T})$ *if and only if* $f = u + \tilde{v}$ *where* $u, v \in \mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$.
(ii) $f \in \mathcal{Q}_{K,0}(\mathbb{T})$ *if and only if* $f = u + \tilde{v}$ *where* $u, v \in \mathcal{Q}_{K,0}(\mathbb{T}) \cap C(\mathbb{T})$.

The paper is organized as follows. Section 2 is devoted to the study of some basic facts about $\mathcal{Q}_K$ spaces. In Section 3 solutions of the $\bar{\partial}$-problem are studied and Theorem 1.1 is proved, while Theorem 1.2 is proved in Section 4. Throught this paper, given a subarc $I \subset \mathbb{T}$ with normalized arclength $|I|$, we denote by $S(I)$ the Carleson box based on I

$$S(I) = \{re^{it} \in \mathbb{D}:\ 1 - |I| < r < 1,\ e^{it} \in I\}.$$

2. Preliminary facts

2.1. Properties of weights

If the weight K satisfies condition (2), we may assume that there exists $c > 0$ such that

$$t^{c-1} K(t) \text{ is decreasing}, \quad 0 < t < \infty. \tag{3}$$

Indeed, in [3] it is proved that if K satisfies (2) then there is a weight K_1 comparable with K satisfying (3). It turns out, also from [3], that if (1) holds, then

$$\int_0^t K(s) \frac{ds}{s} \approx K(t), \quad 0 < t < 1, \tag{4}$$

and if (2) holds, then

$$\int_t^\infty K(s) \frac{ds}{s^2} \approx \frac{K(t)}{t}, \quad t > 0. \tag{5}$$

The following result will be used in Section 3.

Lemma 2.1. *Suppose that* K *satisfies* (1) *and* (2). *Let* $1 - c < \beta \leqslant 1$. *Then for all* $w \in \mathbb{D}$ *with* $1 - |w| < |I|$ *we have*

$$\int_{S(I)} \frac{K(\frac{1-|z|}{|I|})}{(1-|z|)^\beta |1 - \bar{w}z|^2} \, dA(z) \lesssim \frac{K(\frac{1-|w|}{|I|})}{(1-|w|)^\beta}. \tag{6}$$

Proof. Suppose first that $|w| \leqslant 1/2$. Then $|1 - \bar{w}z| \geqslant 1/2$, $z \in \mathbb{D}$, and by (4),

$$\int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^\beta |1 - \bar{w}z|^2} \, dA(z) \leqslant 8\pi \int_0^1 K(s) \frac{ds}{s} < \infty.$$

Since the function $t^{-\beta} K(t)$ is decreasing, an inequality of type (6) holds whenever $|w| \leqslant 1/2$.

Suppose now that $|w| \geqslant 1/2$. Without loss of generality we may assume that I is centered at $e^{i0} = 1$ and $\mathrm{Im}(w) = 0$ and hence that $w = 1 - \alpha$ with $0 < \alpha < |I|$. We split the Carleson box $S(I) = \{z \in \mathbb{D}:\ 1 - |z| < |I|,\ |\arg z| < |I|/2\}$ into $S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \{z:\ 0 < 1 - |z| \leqslant \alpha,\ |\arg z| \leqslant \alpha/2\}, \\ S_2 &= \{z:\ \alpha < 1 - |z| \leqslant |I|,\ |\arg z| \leqslant \alpha/2\}, \\ S_3 &= \{z:\ 0 < 1 - |z| \leqslant |I|,\ \alpha/2 < |\arg z| \leqslant |I|/2\}. \end{aligned}$$

Then, by (4), we have

$$\int_{S_1} \frac{K(\frac{1-|z|}{|I|})}{(1-|z|)^{\beta}|1-\bar{w}z|^2}\,dA(z) \leqslant \alpha \int_0^{\alpha} \frac{K(t/|I|)}{(\alpha+t(1-\alpha))^2}\frac{dt}{t^{\beta}} \leqslant \frac{1}{\alpha}\int_0^{\alpha}\frac{K(t/|I|)}{t^{\beta}}\,dt = \frac{|I|^{1-\beta}}{\alpha}\int_0^{\alpha/|I|}\frac{K(s)}{s^{\beta}}\,ds$$
$$\leqslant \frac{1}{\alpha^{\beta}}\int_0^{\alpha/|I|}\frac{K(s)}{s}\,ds \lesssim \frac{K(\alpha/|I|)}{\alpha^{\beta}},$$

and, since $t^{c-1}K(t)$ is decreasing and $\beta > 1-c$, we have

$$\int_{S_2} \frac{K(\frac{1-|z|}{|I|})}{(1-|z|)^{\beta}|1-\bar{w}z|^2}\,dA(z) \leqslant \alpha \int_{\alpha}^{|I|} \frac{K(t/|I|)}{(\alpha+t(1-\alpha))^2}\frac{dt}{t^{\beta}} \leqslant \frac{1}{(1-\alpha)}\int_{\alpha}^{|I|}\frac{K(t/|I|)}{t^{1+\beta}}\,dt$$
$$\lesssim \frac{K(\alpha/|I|)}{\alpha^{1-c}}\int_{\alpha}^{|I|}\frac{dt}{t^{c+\beta}}\,dt \lesssim \frac{K(\alpha/|I|)}{\alpha^{\beta}},$$

and

$$\int_{S_3} \frac{K(\frac{1-|z|}{|I|})}{(1-|z|)^{\beta}|1-\bar{w}z|^2}\,dA(z) \leqslant 2\int_0^{|I|}\frac{K(t/|I|)}{t^{\beta}}\left(\int_{\alpha}^{|I|}\frac{d\theta}{(\alpha+t(1-\alpha))^2+\sin^2\theta/2}\right)dt \lesssim \int_0^{|I|}\frac{K(t/|I|)}{(\alpha+t(1-\alpha))}\frac{dt}{t^{\beta}}$$
$$\lesssim \frac{1}{\alpha^{\beta}}\int_0^{\alpha}\frac{K(t/|I|)}{t}\,dt + \int_{\alpha}^{|I|}\frac{K(t/|I|)}{t^{1+\beta}}\,dt \lesssim \frac{K(\alpha/|I|)}{\alpha^{\beta}}. \qquad \square$$

2.2. *K-Carleson measures*

For $0 < p < \infty$, we say that a positive measure μ on $\mathbb{D}$ is a p-Carleson measure if

$$\|\mu\|_p = \sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|^p} < \infty,$$

where the supremum is taken over all subarcs I of $\mathbb{T}$. If the right-hand fractions tend to zero as $|I| \to 0$ then μ is said to be a compact p-Carleson measure. Note that the 1-Carleson measures are the classical Carleson measures.

In a similar way, a positive measure μ on $\mathbb{D}$ is said to be a K-Carleson measure if

$$\|\mu\|_K = \sup_{I\subset\mathbb{T}} \mu_K\big(S(I)\big) < \infty,$$

where the supremum is taken over all subarcs I of $\mathbb{T}$, and

$$\mu_K\big(S(I)\big) = \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)d\mu(z).$$

Also, μ is said to be a compact K-Carleson measure if $\|\mu\|_K < \infty$ and

$$\lim_{|I|\to 0} \mu_K\big(S(I)\big) = 0.$$

Clearly, if $K(t) = t^p$, then μ is a K-Carleson measure if and only if the measure $(1-|z|^2)^p\,d\mu(z)$ is a p-Carleson measure. The following result (part (i) proved in [3]) characterizes K-Carleson measures in conformally invariant terms.

Theorem 2.1. *Suppose K satisfies* (1). *Then*

(i) *μ is a K-Carleson measure if and only if*

$$\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)<\infty. \tag{7}$$

(ii) *μ is a compact K-Carleson measure if and only if* (7) *holds and*

$$\lim_{|w|\to 1^-}\int_{\mathbb{D}} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)=0.$$

Proof. Fix $0\neq w\in\mathbb{D}$, and let $I=I_w$ be the arc of center $w/|w|$ and length $(1-|w|)/2\pi$. Then we have

$$\frac{1-|w|^2}{|1-\bar{w}z|^2}\gtrsim\frac{1}{|I|},\qquad z\in S(I).$$

Therefore

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} K\big(1-\left|\varphi_a(z)\right|^2\big)\,d\mu(z)\geqslant\int_{S(I)} K\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2}\right)d\mu(z)\gtrsim\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)d\mu(z),$$

and this is enough to conclude the sufficiency of parts (i) and (ii).

For the converse, let $S_n=S(2^nI)$, where 2^nI denotes the arc with the same center as I and length $2^n|I|$. Then

$$\begin{aligned}
\int_{\mathbb{D}} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)&=\int_{S(I)} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)+\sum_{n=2}^{\infty}\int_{S_n\setminus S_{n-1}} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)\\
&\lesssim\mu_K\big(S(I)\big)+\sum_{n=2}^{\infty}\int_{S_n\setminus S_{n-1}} K\left(\frac{1-|z|}{2^{2n}|I|}\right)d\mu(z)\\
&\lesssim\mu_K\big(S(I)\big)+\sum_{n=2}^{\infty}\sup_{z\in S_n}\frac{K(2^{-2n}(1-|z|)/|I|)}{K(2^{-n}(1-|z|)/|I|)}\int_{S_n} K\left(\frac{1-|z|}{2^n|I|}\right)d\mu(z)\\
&\lesssim\mu_K\big(S(I)\big)+\sum_{n=2}^{\infty}\varphi_K\big(2^{-n}\big)\mu_K(S_n).
\end{aligned}$$

Since $\sum\varphi_K(2^{-n})\approx\int_0^1\varphi_K(s)\frac{ds}{s}<\infty$, if μ is a K-Carleson measure, we obtain that (7) holds.

If μ is a compact K-Carleson measure, then for any $\varepsilon>0$ there exists $\delta>0$ such that the estimate

$$\mu_K\big(S(I)\big)<\varepsilon$$

holds when $|I|<\delta$. Take N such that $\sum_{n\geqslant N}\varphi_K(2^{-n})<\varepsilon$. Then, if $1-|w|<2^{-N}\delta$ and $I=I_w$, we have

$$\begin{aligned}
\int_{\mathbb{D}} K\big(1-\left|\varphi_w(z)\right|^2\big)\,d\mu(z)&\leqslant\mu_K\big(S(I)\big)+\sum_{n=2}^{N-1}\varphi_K\big(2^{-n}\big)\mu_K(S_n)+\sum_{n\geqslant N}\varphi_K\big(2^{-n}\big)\mu_K(S_n)\\
&<\varepsilon\left(1+\sum_{n=1}^{\infty}\varphi_K\big(2^{-n}\big)+\|\mu\|_K\right),
\end{aligned}$$

and this is enough to conclude the desired result. $\square$

Corollary 2.1. *Suppose K satisfies* (1). *Let f be analytic on $\mathbb{D}$. Then*

(i) *$f \in \mathcal{Q}_K$ if and only if $|f'(z)|^2\,dA(z)$ is a K-Carleson measure.*
(ii) *$f \in \mathcal{Q}_{K,0}$ if and only if $|f'(z)|^2\,dA(z)$ is a compact K-Carleson measure.*

Proof. By Theorem 2.1 it is enough to prove that

$$\|f\|^2_{\mathcal{Q}_K} \approx \sup_{w\in\mathbb{D}} \int\limits_{\mathbb{D}} \left|f'(z)\right|^2 K\left(1-\left|\varphi_w(z)\right|^2\right) dA(z).$$

Since $2g(z,w) \geqslant 1-|\varphi_w(z)|^2$, conditions (a) and (e) of K gives

$$\int\limits_{\mathbb{D}} \left|f'(z)\right|^2 K\left(g(z,w)\right) dA(z) \lesssim \int\limits_{\mathbb{D}} \left|f'(z)\right|^2 K\left(1-\left|\varphi_w(z)\right|^2\right) dA(z).$$

To show the remainder estimate, first note that if $|\varphi_w(z)| > 1/4$, we have the reverse inequality $-\log|\varphi_w(z)| \leqslant 4(1-|\varphi_w(z)|^2)$, which yields

$$\int\limits_{|\varphi_w(z)|>1/4} \left|f'(z)\right|^2 K\left(g(z,w)\right) dA(z) \lesssim \int\limits_{\mathbb{D}} \left|f'(z)\right|^2 K\left(1-\left|\varphi_w(z)\right|^2\right) dA(z).$$

If $|z| \leqslant 1/4$, then the subharmonicity of $|f'(z)|^2$ gives

$$\left|\left(f\circ\varphi_w\right)'(z)\right|^2 \lesssim \int\limits_{|\zeta-z|<1/4} \left|(f\circ\varphi_w)'(\zeta)\right|^2 dA(\zeta) \lesssim \int\limits_{\mathbb{D}} \left|(f\circ\varphi_w)'(\zeta)\right|^2 K\left(1-|\zeta|^2\right) dA(\zeta). \tag{8}$$

Hence, using condition (c) of K and (8), we have

$$\begin{aligned}
&\int\limits_{|\varphi_w(z)|<1/4} \left|f'(z)\right|^2 K\left(\log\frac{1}{|\varphi_w(z)|}\right) dA(z) \\
&\quad = \int\limits_{|z|<1/4} \left|(f\circ\varphi_w)'(z)\right|^2 K\left(\log\frac{1}{|z|}\right) dA(z) \\
&\quad \leqslant \left(\int\limits_{\mathbb{D}} \left|(f\circ\varphi_w)'(\zeta)\right|^2 K\left(1-|\zeta|^2\right) dA(\zeta)\right)\left(\int\limits_{|z|<1/4} K\left(\log\frac{1}{|z|}\right) dA(z)\right) \\
&\quad \lesssim \int\limits_{\mathbb{D}} \left|f'(\zeta)\right|^2 K\left(1-\left|\varphi_w(\zeta)\right|^2\right) dA(\zeta). \qquad \square
\end{aligned}$$

Lemma 2.2. *Suppose K satisfies* (2).

(i) *If $d\mu(z) = |f(z)|^2\,dA(z)$ is a K-Carleson measure, then $|f(z)|\,dA(z)$ is a 1-Carleson measure.*
(ii) *If $d\mu(z) = |f(z)|^2\,dA(z)$ is a compact K-Carleson measure, then $|f(z)|\,dA(z)$ is a compact 1-Carleson measure.*

Proof. We know that for some $c>0$, the function $t^{c-1}K(t)$ is decreasing. Then, if $S(I)$ is a Carleson box, by the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\int\limits_{S(I)} \left|f(z)\right| dA(z) &\leqslant \left(\int\limits_{S(I)} \left|f(z)\right|^2 K\left(\frac{1-|z|^2}{|I|}\right) dA(z)\right)^{1/2}\left(\int\limits_{S(I)} \frac{dA(z)}{K((1-|z|^2)/|I|)}\right)^{1/2} \\
&\leqslant \left(\mu_K\left(S(I)\right)\right)^{1/2}\left(\int\limits_{S(I)} \frac{dA(z)}{K((1-|z|^2)/|I|)}\right)^{1/2}
\end{aligned}$$

$$\lesssim \left(\mu_K\left(S(I)\right)\right)^{1/2}\left(|I|^{1-c}\int\limits_{S(I)}\frac{dA(z)}{(1-|z|^2)^{1-c}}\right)^{1/2}$$

$$\lesssim \left(\mu_K\left(S(I)\right)\right)^{1/2}|I|,$$

and this estimate gives (i) and (ii). $\square$

2.3. Boundary valued $\mathcal{Q}_K(\mathbb{T})$ spaces

Let $f\in L^2(\mathbb{T})$. We say that $f\in\mathcal{Q}_K(\mathbb{T})$ if

$$\|f\|^2_{\mathcal{Q}_K(\mathbb{T})}=\sup_{I\subset\mathbb{T}}\int\limits_I\int\limits_I\frac{|f(\zeta)-f(\eta)|^2}{|\zeta-\eta|^2}K\left(\frac{|\zeta-\eta|}{|I|}\right)|d\zeta|\,|d\eta|<\infty,$$

where the supremum is taken over all arcs $I\subset\mathbb{T}$.

If also

$$\lim_{|I|\to 0}\int\limits_I\int\limits_I\frac{|f(\zeta)-f(\eta)|^2}{|\zeta-\eta|^2}K\left(\frac{|\zeta-\eta|}{|I|}\right)|d\zeta||d\eta|=0,$$

then we say that f is in $\mathcal{Q}_{K,0}(\mathbb{T})$. From Theorem 4.1 of [3], if K satisfies (1) and (2), then a function $f\in H^2$ is in $\mathcal{Q}_K$ if and only if it has boundary values in $\mathcal{Q}_K(\mathbb{T})$. Given $f\in L^1(\mathbb{T})$ let $\hat{f}$ be its Poisson extension, that is

$$\hat{f}(z)=\frac{1}{2\pi}\int\limits_0^{2\pi}f\left(e^{i\theta}\right)P_z(\theta)\,d\theta,$$

where

$$P_z(\theta)=\frac{1-|z|^2}{|e^{i\theta}-z|^2}.$$

Then we will obtain the following theorem which can be viewed as an extension of Corollary 2.1.

Theorem 2.2. *Suppose that* (1) *and* (2) *hold for K, and let $f\in L^2(\mathbb{T})$. Then the following conditions are equivalent*:

(i) $f\in\mathcal{Q}_K(\mathbb{T})$;
(ii) $|\nabla\hat{f}(z)|^2\,dA(z)$ *is a K-Carleson measure.*

Proof. This result was essentially done in [3]. Nevertheless, we provide a sketch of the proof. Let $f\in L^2(\mathbb{T})$, and suppose that $f\in\mathcal{Q}_K(\mathbb{T})$. Using the fact that $K(t)/t$ is decreasing we have

$$\frac{1}{|I|^2}\int\limits_I\int\limits_I\left|f\left(e^{it}\right)-f\left(e^{i\theta}\right)\right|^2dt\,d\theta\leqslant\frac{1}{|I|}\int\limits_I\int\limits_I\frac{|f(e^{it})-f(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|}\,dt\,d\theta$$

$$\lesssim\int\limits_I\int\limits_I\frac{|f(e^{it})-f(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}K\left(\frac{|e^{it}-e^{i\theta}|}{|I|}\right)dt\,d\theta\lesssim\|f\|_{\mathcal{Q}_K(\mathbb{T})}.$$

Hence $f\in BMO(\mathbb{T})$. Let $S(I)$ be a Carleson box. Without loss of generality we can assume that I is centered at 1. Let J be another arc with the same center as I and $|J|=3|I|$. From Lemma 4.4 of [3] we have

$$\int\limits_{S(I)}\left|\nabla\hat{f}(z)\right|^2K\left(\frac{1-|z|}{|I|}\right)dA(z)\lesssim\int\limits_J\int\limits_J\frac{|f(e^{it})-f(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}K\left(\frac{|e^{it}-e^{i\theta}|}{|I|}\right)dt\,d\theta$$

$$+|I|^2\left(\int\limits_{e^{it}\notin I}\frac{|f(e^{it})-f_J|}{t^2}\,dt\right)^2,$$

where $f_J = |J|^{-1} \int_J f(e^{i\theta})\,d\theta$. So, it is enough to show that if $f \in BMO(\mathbb{T})$, then

$$\int\limits_{|t| \geqslant |J|/3} \left|f\left(e^{it}\right) - f_J\right| \frac{dt}{t^2} \lesssim \frac{1}{|I|}$$

and this estimate can be found, for example, in [4].

For the converse, suppose now that (ii) holds. Then the result will follow from the fact that if F is a C^1 function on $\overline{\mathbb{D}}$ with $F|_{\mathbb{T}} = f$, then

$$\int\limits_I \int\limits_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^2} K\left(\frac{|\zeta - \eta|}{|I|}\right) |d\zeta||d\eta| \lesssim \int\limits_{S(I)} \left|\nabla F(z)\right|^2 K\left(\frac{1-|z|}{|I|}\right) dA(z). \tag{9}$$

For a proof of this estimate we refer to [3]. □

Some consequences of the estimate (9) are the following ones.

Corollary 2.2. *Let F be a C^1 function defined on $\overline{\mathbb{D}}$ such that $F|_{\mathbb{T}} = f$.*

(a) *If $|\nabla F(z)|^2\,dA(z)$ is a K-Carleson measure, then $f \in \mathcal{Q}_K(\mathbb{T})$.*
(b) *If $|\nabla F(z)|^2\,dA(z)$ is a compact K-Carleson measure, then $f \in \mathcal{Q}_{K,0}(\mathbb{T})$.*

Corollary 2.3. *Let $f \in \mathcal{Q}_K(\mathbb{T})$. Then the following conditions are equivalent*:

(i) $f \in \mathcal{Q}_{K,0}(\mathbb{T})$;
(ii) $\lim_{t\to 0} \|R_t f - f\|_{\mathcal{Q}_K(\mathbb{T})} = 0$, *where* $R_t f(e^{i\theta}) = f(e^{i(\theta - t)})$;
(iii) $\lim_{r\to 1} \|f_r - f\|_{\mathcal{Q}_K}(\mathbb{T}) = 0$, *where* $f_r(e^{i\theta}) = \hat{f}(re^{i\theta})$.

Proof. (i) ⇒ (ii). If $f \in \mathcal{Q}_{K,0}(\mathbb{T})$ and $F_t = R_t f - f$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that the estimate

$$\int\limits_I \int\limits_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} K\left(\frac{|e^{i\theta} - e^{i\varphi}|}{|I|}\right) d\theta\,d\varphi < \varepsilon$$

holds when $|I| < \delta$. Therefore, for any arc I with $|I| < \delta$ one has

$$\begin{aligned}
&\int\limits_I \int\limits_I \frac{|F_t(e^{i\theta}) - F_t(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} K\left(\frac{|e^{i\theta} - e^{i\varphi}|}{|I|}\right) d\theta\,d\varphi \\
&\quad\lesssim \int\limits_I \int\limits_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} K\left(\frac{|e^{i\theta} - e^{i\varphi}|}{|I|}\right) d\theta\,d\varphi \\
&\qquad + \int\limits_I \int\limits_I \frac{|f(e^{i(\theta-t)}) - f(e^{i(\varphi-t)})|^2}{|e^{i(\theta-t)} - e^{i(\varphi-t)}|^2} K\left(\frac{|e^{i(\theta-t)} - e^{i(\varphi-t)}|}{|I|}\right) d\theta\,d\varphi \lesssim \varepsilon.
\end{aligned}$$

If I is an arc with $|I| \geqslant \delta$, then using (9) we have

$$\begin{aligned}
&\int\limits_I \int\limits_I \frac{|F_t(e^{i\theta}) - F_t(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} K\left(\frac{|e^{i\theta} - e^{i\varphi}|}{|I|}\right) d\theta\,d\varphi \\
&\quad\lesssim \int\limits_{S(I)} \left|\nabla \hat{F}_t(z)\right|^2 K\left(\frac{1-|z|}{|I|}\right) dA(z) \lesssim \int\limits_{\mathbb{D}} \left|\nabla\left(\hat{f}\left(ze^{-it}\right) - \hat{f}(z)\right)\right|^2 K\left(\frac{1-|z|}{\delta}\right) dA(z),
\end{aligned}$$

and this tends to zero as $t \to 0$. Therefore (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose that $f \in \mathcal{Q}_K(\mathbb{T})$ satisfies (ii). We use Minkowski inequality (see [8, p. 271]) in

$$f\left(e^{i\theta}\right) - f_r\left(e^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \left(f\left(e^{i\theta}\right) - f\left(e^{i(\theta-\varphi)}\right)\right) P_r(\varphi)\, d\varphi$$

where $P_r(\varphi) = \frac{1-r^2}{1-2r\cos\varphi + r^2}$ is the Poisson kernel, to get that for any small $\varepsilon > 0$,

$$\begin{aligned}\|f - f_r\|_{\mathcal{Q}_K(\mathbb{T})} &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \|f - R_\varphi f\|_{\mathcal{Q}_K(\mathbb{T})} P_r(\varphi)\, d\varphi \\ &\lesssim \int_{|\varphi|<\varepsilon} \|f - R_\varphi f\|_{\mathcal{Q}_K(\mathbb{T})} P_r(\varphi)\, d\varphi + \int_{|\varphi|\geqslant\varepsilon} \|f\|_{\mathcal{Q}_K(\mathbb{T})} P_r(\varphi)\, d\varphi,\end{aligned}$$

and this estimate gives (iii).

(iii) $\Rightarrow$ (i). This implication is obvious since $f_r \in \mathcal{Q}_{K,0}(\mathbb{T})$ and $\mathcal{Q}_{K,0}(\mathbb{T})$ is closed in $\mathcal{Q}_K(\mathbb{T})$. $\square$

Proposition 2.1. *Suppose that K satisfies* (1) *and* (2). *If $f \in \mathcal{Q}_K(\mathbb{T})$ then $\tilde{f} \in \mathcal{Q}_K(\mathbb{T})$.*

Proof. From [3, Theorem 6.1] $f \in \mathcal{Q}_K(\mathbb{T})$ if and only if

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \left(\widehat{|f|^2}(z) - \left|\hat{f}(z)\right|^2\right)\tilde{K}\left(\left|\varphi_w(z)\right|^2\right)\left|\varphi_w'(z)\right|^2\, dA(z) < \infty,$$

where

$$\tilde{K}\left(|z|^2\right) = \frac{\partial^2 K(1-|z|^2)}{\partial z \partial \bar{z}}, \quad z \in \mathbb{D}.$$

Since

$$\widehat{|f|^2}(z) - \left|\hat{f}(z)\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{i\theta}\right) - \hat{f}(z)\right|^2 P_z(\theta)\, d\theta,$$

the result follows from the fact that any $f \in BMO(\mathbb{T})$ enjoys the identity

$$\int_0^{2\pi} \left|f\left(e^{i\theta}\right) - \hat{f}(z)\right|^2 P_z(\theta)\, d\theta = \int_0^{2\pi} \left|\tilde{f}\left(e^{i\theta}\right) - \tilde{f}(z)\right|^2 P_z(\theta)\, d\theta. \qquad \square$$

Remark. Following the proof of Theorem 6.1 in [3], one can see that $f \in \mathcal{Q}_{K,0}(\mathbb{T})$ if and only if $f \in \mathcal{Q}_K(\mathbb{T})$ and

$$\lim_{|w|\to 1^-} \int_{\mathbb{D}} \left(\widehat{|f|^2}(z) - \left|\hat{f}(z)\right|^2\right)\tilde{K}\left(\left|\varphi_w(z)\right|^2\right)\left|\varphi_w'(z)\right|^2\, dA(z) = 0.$$

Therefore, if K satisfies (1) and (2), we have that $\tilde{f} \in \mathcal{Q}_{K,0}(\mathbb{T})$ if $f \in \mathcal{Q}_{K,0}(\mathbb{T})$.

Let P denote the Szegö projection from $L^2(\mathbb{T})$ onto H^2

$$Pf(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}}\, d\theta, \quad f \in L^2(\mathbb{T}),\ z \in \mathbb{D}.$$

Corollary 2.4. *Suppose that K satisfies* (1) *and* (2). *Then $P : \mathcal{Q}_K(\mathbb{T}) \to \mathcal{Q}_K$ is bounded and surjective.*

Proof. An easy calculation shows that $\hat{f}(z) + i\tilde{f}(z) = 2(Pf)(z) - \hat{f}(0)$, and this gives the boundedness of P by Proposition 2.1. Since $Pf = f$ whenever $f \in \mathcal{Q}_K$, we have that P is onto and the proof is complete. □

Remark. In a similar way, if K satisfies (1) and (2), then $P : \mathcal{Q}_{K,0}(\mathbb{T}) \to \mathcal{Q}_{K,0}$ is bounded and surjective.

3. The $\bar{\partial}$-equation and the corona problem

Given a 1-Carleson measure μ on the unit disc, it is well known (see [5]) that the $\bar{\partial}$-problem $\bar{\partial}F = \mu$, has a solution F, in the sense of distributions, satisfying $\|F\|_{L^\infty(\mathbb{T})} \leqslant C\|\mu\|_1$. In [6], P. Jones found that such a solution F can be given by a simple and flexible formula,

$$F(z) = \int_{\mathbb{D}} K_\mu(z,\zeta)\,d\mu(\zeta), \tag{10}$$

where

$$K_\mu(z,\zeta) = \frac{1-|\zeta|^2}{\pi(1-\bar{\zeta}z)(z-\zeta)} \exp\left\{ \int_{|w|\geqslant|\zeta|} \left(\frac{1+\bar{w}\zeta}{1-\bar{w}\zeta} - \frac{1+\bar{w}z}{1-\bar{w}z} \right) \frac{d\mu(w)}{\|\mu\|_1} \right\}.$$

The estimate $\int_{\mathbb{D}} |K_\mu(e^{i\theta},\zeta)|\,d\mu(\zeta) \leqslant C_1\|\mu\|_1$ shows that $F \in L^\infty(\mathbb{T})$. Therefore, if $|g(z)|\,dA(z)$ is a 1-Carleson measure, then the equation $\bar{\partial}F = g$ has a solution $F \in L^\infty(\mathbb{T})$. We are interested on some similar result for $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$. In order to find a solution in $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ we need first the following result.

Lemma 3.1. *Suppose that K satisfies* (1) *and* (2). *Let*

$$Tf(z) = \int_{\mathbb{D}} \frac{f(w)}{|1-\bar{w}z|^2}\,dA(w).$$

(i) *If $d\mu(z) = |f(z)|^2\,dA(z)$ is a K-Carleson measure, then $d\nu(z) = |Tf(z)|^2\,dA(z)$ is also a K-Carleson measure and $\|\nu\|_K \lesssim \|\mu\|_K^{1/2}$.*
(ii) *If $d\mu(z) = |f(z)|^2\,dA(z)$ is a compact K-Carleson measure, then $d\nu(z) = |Tf(z)|^2\,dA(z)$ is also a compact K-Carleson measure.*

Proof. For the Carleson box $S(I)$, we have

$$\begin{aligned}
\nu_K\big(S(I)\big) &= \int_{S(I)} \big|Tf(z)\big|^2 K\left(\frac{1-|z|}{|I|}\right) dA(z) \\
&\leqslant \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) \left\{ \left(\int_{S(2I)} + \int_{\mathbb{D}\setminus S(2I)} \right) \frac{|f(w)|}{|1-\bar{w}z|^2}\,dA(w) \right\}^2 dA(z) \\
&\lesssim \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) \left(\int_{S(2I)} \frac{|f(w)|}{|1-\bar{w}z|^2}\,dA(w) \right)^2 dA(z) \\
&\quad + \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) \left(\int_{\mathbb{D}\setminus S(2I)} \frac{|f(w)|}{|1-\bar{w}z|^2}\,dA(w) \right)^2 dA(z) \\
&= I_1 + I_2.
\end{aligned}$$

To estimate I_1, we use Schur's lemma (see [11, p. 42]). Indeed, consider

$$k(z,w) = \frac{\{K_I(1-|z|)\}^{1/2}\{K_I(1-|w|)\}^{-1/2}}{|1-\bar{w}z|^2},$$

where $K_I(t) = K(t/|I|)$, and its induced integral operator on $S(2I)$

$$T_k f(z) = \int_{S(2I)} f(w)k(z,w)\,dA(w).$$

Taking $\beta \in (1-c, 1)$ and applying Lemma 2.1, we get

$$\int_{S(2I)} k(z,w)\frac{\{K_I(1-|w|)\}^{1/2}}{(1-|w|)^\beta} \lesssim \frac{\{K_I(1-|z|)\}^{1/2}}{(1-|z|)^\beta}$$

and

$$\int_{S(2I)} k(z,w)\frac{\{K_I(1-|z|)\}^{1/2}}{(1-|z|)^\beta} \lesssim \frac{\{K_I(1-|w|)\}^{1/2}}{(1-|w|)^\beta}.$$

Therefore the operator T_k is bounded from $L^2(S(2I))$ to $L^2(S(2I))$. Consider the function $g(w) = \{K_I(1-|w|)\}^{1/2}|f(w)|1_{S(2I)}(w)$, then we have

$$I_1 \lesssim \int_{S(2I)} \big(T_k g(z)\big)^2\,dA(z) \lesssim \int_{S(2I)} \big|g(z)\big|^2\,dA(z) = \int_{S(2I)} \big|f(z)\big|^2 K\left(\frac{1-|z|}{|I|}\right)dA(z) \lesssim \mu_K\big(S(2I)\big).$$

Since $d\mu(z) = |f(z)|^2\,dA(z)$ is a K-Carleson measure then, by Lemma 2.2, $d\mu_1(z) = |f(z)|\,dA(z)$ is a 1-Carleson measure with $\|\mu_1\|_1 \lesssim \|\mu\|_K^{1/2}$. This deduces

$$I_2 \lesssim \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)\left(\sum_{n=1}^{\infty}\int_{S(2^{n+1}I)\setminus S(2^nI)} \frac{|f(w)|}{|1-\bar{w}z|^2}\,dA(w)\right)^2 dA(z)$$

$$\lesssim \|\mu\|_K^{1/2}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)\left(\sum_{n=1}^{\infty}\frac{2^{n+1}|I|}{(2^n|I|)^2}\right)^2 dA(z) \lesssim \|\mu\|_K^{1/2},$$

and therefore (i) holds.

If $d\mu(z) = |f(z)|^2\,dA(z)$ is a compact K-Carleson measure, then $|f(z)|\,dA(z)$ is a compact 1-Carleson measure by Lemma 2.2. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{S(J)} \big|f(z)\big|\,dA(z) < \varepsilon|J|$$

for each arc $J \subset \mathbb{T}$ with $|J| < \delta$. Take N so that $\sum_{n\geqslant N} 2^{-n} < \varepsilon$. Then, if $|I| < 2^{-N}\delta$, we can refine the previous estimate to obtain

$$I_2 \lesssim \int_{S(I)} \left(\sum_{n=1}^{N-1} 2^{-2n}|I|^{-2}\int_{S(2^{n+1}I)} \big|f(w)\big|\,dA(w) + \sum_{n\geqslant N} 2^{-n}|I|^{-1}\right)^2 dA(z)$$

$$\lesssim |I|^2\left(\sum_{n=1}^{N-1} 2^{-n}|I|^{-1}\varepsilon + \varepsilon|I|^{-1}\right)^2 \leqslant 4\varepsilon^2,$$

and this is enough to obtain (ii). $\square$

Theorem 3.1. *Suppose that K satisfies* (1) *and* (2). *If $d\lambda(z) = |g(z)|^2\,dA(z)$ is a* (*compact*) *K-Carleson measure, then there is a function f defined on $\overline{\mathbb{D}}$ with boundary values in $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$* (*in $\mathcal{Q}_{K,0}(\mathbb{T}) \cap L^\infty(\mathbb{T})$*) *such that*

$$\bar{\partial} f(z) = g(z), \quad z \in \mathbb{D}.$$

Actually, $\|f\|_{\mathcal{Q}_K(\mathbb{T})} + \|f\|_{L^\infty(\mathbb{T})} \lesssim \|\lambda\|_K^{1/2}$.

Proof. By Lemma 2.2, the measure $d\mu(z) = |g(z)|\,dA(z)$ is a 1-Carleson measure. Thus the function F given by (10) is in $L^\infty(\mathbb{T})$ and satisfies $\bar\partial F = g$. But we want also $F \in \mathcal{Q}_K(\mathbb{T})$. For this purpose, define a new function G on $\mathbb{D}$ with the same boundary values as $zF(z)$ on $\mathbb{T}$,

$$G(z) = \frac{i}{\pi} \int_{\mathbb{D}} \frac{1-|\zeta|^2}{|1-\bar\zeta z|^2} \exp\left\{ \int_{|w|\geqslant|\zeta|} \left(\frac{1+\bar w\zeta}{1-\bar w\zeta} - \frac{1+\bar w z}{1-\bar w z}\right)|g(w)| \frac{dA(w)}{\|\mu\|_1}\right\} g(\zeta)\,dA(\zeta).$$

By Corollary 2.2, it is enough to show that $|\nabla G(z)|^2\,dA(z)$ is a (compact) K-Carleson measure. Without loss of generality we may assume that $g(z) \geqslant 0$ and $\|\mu\|_1 = 1$. Then

$$\operatorname{Re}\left(\int_{|w|\geqslant|\zeta|} \frac{1+\bar w\zeta}{1-\bar w\zeta} g(w)\,dA(w)\right) \leqslant 2 \int_{\mathbb{D}} \frac{1-|\zeta|^2}{|1-\bar w\zeta|^2} g(w)\,dA(w) \leqslant C_2 \tag{11}$$

where $C_2 > 0$ is a constant independent of $\zeta \in \mathbb{D}$. Moreover,

$$\int_{\mathbb{D}} \frac{1-|\zeta z|^2}{|1-\bar\zeta z|^2} \exp\left\{ - \int_{|w|\geqslant|\zeta|} \frac{1-|wz|^2}{|1-\bar w z|^2} g(w)\,dA(w)\right\} g(\zeta)\,dA(\zeta) \leqslant 1 \tag{12}$$

(see the proof of Lemma 2.1 in [6]). Using (11) and (12) one can show that

$$\left|\nabla G(z)\right| \lesssim \int_{\mathbb{D}} \frac{g(w)}{|1-\bar w z|^2}\,dA(w).$$

Since $|g(w)|^2\,dA(w)$ is a (compact) K-Carleson measure, we can apply Lemma 3.1 to obtain that $|\nabla G(z)|^2\,dA(z)$ is a (compact) K-Carleson measure and this completes the proof. □

Proof of Theorem 1.1. From a standard normal families argument, we can assume that the given functions $f_1, \ldots, f_n$ are analytic on a neighbourhood of the closed unit disc $\overline{\mathbb{D}}$. It is clear that

$$h_j(z) = \overline{f_j(z)} \Big/ \left(\sum_{k=1}^{n} \left|f_k(z)\right|^2 \right)$$

are nonanalytic functions making

$$\sum_{k=1}^{n} f_k h_k = 1.$$

As in the case of H^∞ (see [5]), in order to replace h_k by functions in $\mathcal{Q}_K \cap H^\infty$ (in $\mathcal{Q}_{K,0} \cap H^\infty$) one needs to solve the $\bar\partial$ equations

$$\bar\partial b_{j,k} = h_j\,\bar\partial h_k, \quad 1 \leqslant j,k \leqslant n,$$

with solutions in $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ (in $\mathcal{Q}_{K,0} \cap H^\infty$). To do this it is enough to deal with an equation $\bar\partial b = h$ where $h = h_j \bar\partial h_k$. An easy calculation shows

$$\left|h(z)\right| \leqslant C \sum_{k=1}^{n} \left|f_k'(z)\right|.$$

Hence, by Corollary 2.1, $|h(z)|^2\,dA(z)$ is a (compact) K-Carleson measure. By Theorem 3.1, there is a solution $b \in \mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ (in $\mathcal{Q}_{K,0}(\mathbb{T}) \cap L^\infty(\mathbb{T})$) of the equation $\bar\partial b = h$ such that

$$\|b\|_{\mathcal{Q}_K(\mathbb{T})} + \|b\|_{L^\infty(\mathbb{T})} \leqslant C \sum_{k=1}^{n} \|f_k\|_{\mathcal{Q}_K}. \qquad \square$$

4. Fefferman–Stein type decomposition

Another application of Theorem 3.1 is a decomposition of $\mathcal{Q}_K(\mathbb{T})$ similar to the Fefferman–Stein decomposition of $BMO(\mathbb{T})$.

Proof of Theorem 1.2. (i) If $f = u + \tilde{v}$ with $u, v \in \mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$, then it follows from Proposition 2.1 that $\tilde{v} \in \mathcal{Q}_K(\mathbb{T})$ and hence $f \in \mathcal{Q}_K(\mathbb{T})$.

For the converse, it is enough to consider the case that $f \in \mathcal{Q}_K(\mathbb{T})$ is real-valued. We find immediately that $F = f + i\tilde{f} \in \mathcal{Q}_K(\mathbb{T})$ and its harmonic extension $\hat{F}$ is in $\mathcal{Q}_K$. From Theorem 2.1, one has that $|\nabla \hat{F}(z)|^2\, dA(z)$ is a K-Carleson measure, and then $|\bar{\partial} f(z)|^2\, dA(z)$ is a K-Carleson measure. By Theorem 3.1 there is a function $g \in \mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ such that $\bar{\partial} g = \bar{\partial} f$. Hence the function $h = f - g$ is analytic on $\mathbb{D}$ and $g \in \mathcal{Q}_K$. Put $u = \operatorname{Re} g$, then $f - u = -\widetilde{\operatorname{Im} g}$. Therefore we have that $f = u + \tilde{v}$, where $u = \operatorname{Re} g$ and $v = -\operatorname{Im} g$ belong to $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$.

(ii) Let $f \in \mathcal{Q}_{K,0}(\mathbb{T})$. From (i) it follows that $f = u_1 + \tilde{u}_2$ for some functions $u_1, u_2 \in \mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ and

$$\|u_j\| = \|u_j\|_{L^\infty(\mathbb{T})} + \|u_j\|_{\mathcal{Q}_K(\mathbb{T})} \leqslant C\|f\|_{\mathcal{Q}_K(\mathbb{T})}, \quad j = 1, 2,$$

where $C > 0$ is a constant independent of f and u_j. By Corollary 2.3 there is an $r \in (0,1)$ such that $\|f - f_r\|_{\mathcal{Q}_K(\mathbb{T})} \leqslant \|f\|_{\mathcal{Q}_K(\mathbb{T})}/2$. Let $u_1^{(1)} = (u_1)_r$ and $u_2^{(1)} = (u_2)_r$. Then $u_j^{(1)} \in \mathcal{Q}_{K,0}(\mathbb{T}) \cap C(\mathbb{T})$ and $f_r = u_1^{(1)} + \widetilde{u_2^{(1)}}$, so that

$$\left\| f - \left(u_1^{(1)} + \widetilde{u_2^{(1)}}\right)\right\|_{\mathcal{Q}_K(\mathbb{T})} = \|f - f_r\|_{\mathcal{Q}_K(\mathbb{T})} \leqslant \frac{\|f\|_{\mathcal{Q}_K(\mathbb{T})}}{2}.$$

Hence the function $F_1 = f - (u_1^{(1)} + \widetilde{u_2^{(1)}}) = u_1 - u_1^{(1)} + \widetilde{u_2 - u_2^{(1)}}$ is in $\mathcal{Q}_{K,0}(\mathbb{T})$ and $\|F_1\|_{\mathcal{Q}_K(\mathbb{T})} \leqslant \|f\|_{\mathcal{Q}_K(\mathbb{T})}/2$. Repeating the above argument with F_1 and iterating, we obtain $f = u + \tilde{v}$ where

$$u = \sum_{k=1}^{\infty} u_1^{(k)} \quad \text{and} \quad v = \sum_{k=1}^{\infty} u_2^{(k)}$$

with $u_1^{(k)}, u_2^{(k)} \in \mathcal{Q}_K(\mathbb{T}) \cap C(\mathbb{T})$ and

$$\sum_k \|u_1^{(k)}\| + \sum_k \|u_2^{(k)}\| \leqslant 4C\|f\|_{\mathcal{Q}_K(\mathbb{T})}.$$

That proves (ii). □

As a consequence of this decomposition of $\mathcal{Q}_K(\mathbb{T})$, we obtain the following improvement of Corollary 2.4.

Corollary 4.1. *Suppose that* (1) *and* (2) *hold for K. Then the Szegö projection P maps $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ onto $\mathcal{Q}_K$.*

Proof. It is enough to show that P is onto. By Theorem 1.2, if $f \in \mathcal{Q}_K$ then there are functions g, h in $\mathcal{Q}_K(\mathbb{T}) \cap L^\infty(\mathbb{T})$ such that $f = g + \tilde{h}$. Therefore

$$f = Pf = Pg + P\tilde{h} = Pg + \frac{i\tilde{\tilde{h}} + \tilde{h} - \hat{\tilde{h}}(0)}{2} = P(g - ih) + \hat{h}(0) - \hat{\tilde{h}}(0),$$

concluding the proof. □

Remark. In a similar way, one can show that if K satisfies (1) and (2), then the Szegö projection maps $\mathcal{Q}_{K,0}(\mathbb{T}) \cap C(\mathbb{T})$ onto $\mathcal{Q}_{K,0}$.

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